Total Pages : 4 UM(I)–M(2)

Paper II

95

2013

Full Marks : 100

Time : 3 hours

Answer any **ten** questions

The questions are of equal value

Candidates are required to give their answers in their own words as far as practicable.

1. If a function $f(z)$ is analytic within and on a positively oriented closed contour C then for every point a within the interior of C, prove that

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2}\, dz.$$

2. State and prove maximum modulus theorem.

3. Let $f(z)$ be analytic in the entire complex plane. suppose that $\frac{f(z)}{z^k}$, for some non-negative integer k, remains bounded as $z \to \infty$. Then prove that $f(z)$ is a polynomial of degree not exceeding k.

(*Turn Over*)

4. If $f(z)$ is analytic in $|z-a|<R$, then prove that 10

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots.$$

$$+\frac{(z-a)^n}{n!}f^n(a)+\ldots, \forall z, |z-a|<R.$$

Also show that this power series representation of $f(z)$ in $(z-a), \forall z, |z-a|<R$ is unique.

5. Find the domains of convergence of following series :

(i) $\sum_{n=1}^{\infty} n^2\left(\frac{z^2+1}{1+i}\right)^n$

(ii) $\sum_{n=1}^{\infty} \frac{1.3.5.....(2n-1)}{n!}\left(\frac{1-z}{z}\right)^n$

6. Let a be an essential singularity of $f(z)$. Let c be any given complex number. Then for any given $\varepsilon>0, r>0, \exists \xi$ in $|z-a|<r$ such that $|f(\xi)-c|<\varepsilon$.

7. Employing contour integration, prove that 10

$$\int_0^{2\pi} \frac{d\theta}{1+a^2-a\left(e^{i\theta}+e^{-i\theta}\right)}=\frac{2\pi}{1-a^2}, 0<a<1.$$

8. Evaluate $\int_0^{\infty} \frac{dx}{x^4+a^4}$, $a>0$. 10

9. Show that the set of all bilinear transformations forms a non-abelian group under the product of transformations.

10. Find the bilinear transformation which maps the points $z_1=2$, $z_2=i$ and $z_3=-2$ into the points $w_1=1$, $w_2=i$ and $w_3=-1$. 10

11. Write down the power series method of analytic continuation. 10

12. Show that the series :

(i) $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$

(ii) $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$

are analytic continuations of each other.

13. Show that the function

$$f(z) = \frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \frac{z^3}{a^4} + \dots\dots$$

can be continued analytically.

14. State and prove Casorati-Weierstrass theorem.

15. Show, by Rouche's theorem, that the equation $z^5 + 15z + 1 = 0$ has one root in the disc $|z| < \frac{3}{2}$ and four roots in the annulus $\frac{3}{2} < |z| < 2$.
