2013

Full Marks : 100

Paper IV
94

Time : 3 hours

Answer any **ten** questions

The figures in the right-hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable.

1. Define isomorphism of groups. If f be an isomorphic mapping of a group G on to G'; prove the following :

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   (*a*) The f image of identity e of G is identity of G'.

   (*b*) The order of an element a of G is equal to the order of its image $f(a)$. $2+4+4=10$

2. Define homomorphism of groups both into and onto. Prove that every homomorphic image of a group G is isomorphic to some quotient group of G. $2+8=10$

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(*Turn Over*)

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3. State and prove Chinese remainder theorem. 10

4. Solve the system of linear congruence : 10

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$$x \equiv 3 \pmod{11}$$
$$x \equiv 5 \pmod{19}$$
$$x \equiv 10 \pmod{29}$$

5. Evaluate $\tau(3000)$ and $\sigma(3000)$. Prove that Möbius function $\mu(n)$ is multiplicative i.e

$\mu(m.n) = \mu(m).\mu(n), m$ and n are relatively primes. $2+2+6=10$

6. If p and q are distinct odd primes, prove that

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \begin{cases} 1 \text{ if } p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4} \\ -1 \text{ if } p \equiv q \equiv 3 \pmod{4} \end{cases}$$

Evaluate $\left(\frac{59}{131}\right)$. $8+2=10$

7. Prove that an odd prime p is expressible as a sum of two squares iff $p \equiv 1 \pmod{4}$.

Express 1225 as a sum of two squares. $8+2=10$

8. Define conjugate elements of a group G. Prove that relation of conjugacy is an equivalence relation on G. 10 $8+2=10$

9. Define self-conjugate element of a group G. Prove that centre z of a group G is normal subgroup of G. 10 $2+8=10$

10. If $o(G) = p^n$ where p is prime number then the centre $z \neq \{e\}$, e being identity of group G. 10

11. State and prove Sylow's first theorem. 10 10

12. State Sylow's second theorem. Prove that if H is a p-Sylow subgroup of G and $x \in G$ then $x^{-1}Hx$ is also a p-Sylow subgroup of G. $2+8=10$

13. Under what condition a group G is said solvable ? 8 Prove that a subgroup of solvable group is solvable. $2+8=10$

14. Show that symmetric group P_3 of degree 3 is solvable. If G is a group and N is a normal subgroup of G such that both N and G/N are solvable then G is also solvable. $2+8=10$

15. Prove that a group G is the direct product of its two subgroups H and K iff : 10

1. H and K are normal subgroups of G
2. $H \cap K = \{e\}$
3. $G = HK$. 10

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